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Some Contraction on G-Banach Space

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Abstract - In this paper we prove some results of fixed point theorems in G- Banach space.Our result are version of some known results in ordinary Banach Spaces.

Keywords : *Fixed point, Common Fixed point, G-Banach space ,Continuous Mapping, Weakly Compatible Mappings.*

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Strictly as per the compliance and regulations of :

Some Contraction on G- Banach Space

Ramakant Bhardwaj

Abstract - In this paper we prove some results of fixed point theorems in G - Banach space.Our result are version of some known results in ordinary Banach Spaces.

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I. Introduction & Prilimnaries

This is well known that, Banach contraction principle is the fundamental contraction principle for proving fixed point results. The concept of G- Banach space is introduce by [11], which is a probable modification of the ordinary Banach Space. In this section some properties about G- Banach space are recall. In section 2, fixed point and common fixed point theorems for four weakly compatible maps in G- Banach space are proved .

In what follows, N *be the set of natural numbers and* R^+ *be the set of all positive real numbers.*

Let binary operation $\nabla : R^+ \times R^+ \to R^+$ *satisfies the following conditions:*

i. ∇ *is associative and commutative,*

ii. ∇ *is continuous.*

Five typical examples are as follows:

for each $a, b \in R^+$

I. $a \nabla b = max\{a, b\}$

II. $a \nabla b = a + b$

III. $a \nabla b = a.b$

IV. $a \nabla b = a.b + a + b$

V. $a \nabla b = \frac{ab}{max\{a,b,1\}}$

Definition: 1.1 [11]

The binary operation ∇ *is said to satisfy* α *– property if there exists a positive real number* α*, such that* $a \nabla b \le max\{a, b\}$ *for every* $a, b \in R^+$

Example: 1.2

If we define $a \nabla b = a + b$*, for each* $a, b \in R^+$*, then for* $\alpha \ge 2$*, we have*

$$a \nabla b \le \alpha \; max\{a, b\}$$

If we define $a \nabla b = \frac{ab}{max\{a,b,1\}}$*, for each* $a, b \in R^+$*, then for* $\alpha \ge 1$*, we have*

$$a \nabla b \le \alpha \; max\{a, b\}$$

Definition:- 1.3[11]

Let X be a nonempty set , A Generalized Normed Space on X, is a function $\|.\|_g : X \times X \to R^+$*, that satisfies the following conditions for each* $x, y, z \in X$.

(1) $\|x - y\|_g > 0$

(2) $\|x - y\|_g = 0$ *if and only if* $x = y$

(3) $\|x - y\|_g = \| y - x\|_g$

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(4) $\|\alpha x\|_g = |\alpha| \|x\|_g$ *for any scalar* α.

(5) $\| x-y\|_g \le \|x-z\|_g \,\nabla\, \|z-x\|_g$

The pair $(X, \|.\|_g)$ *is called Generalized Normed Space, or simply* G*- Normed Space.*

Definition: - 1.4[11]

A sequence $\{x_n\}$ *in* X *is said converges to* x*, if* $\|x_n - x\|_g \to 0$ *, as* $n \to \infty$*. That is for each* $\epsilon > 0$ *there exists* $n_0 \in N$ *such that, for every* $n \ge n_0$ *implies that,* $\|x_n - x\|_g < \epsilon$*.*

Definition:- 1.5 [11]

A sequence $\{x_n\}$ *is said to be Cauchy sequence if for every* $\epsilon > 0$*, there exists* $n_0 \in N$ *such that* $\|x_m - x_n\|_g < \epsilon$ *for each* $m,n \ge n_0$*.* G*- Normed space is said to be* G*- Banach space if every Cauchy sequence is converges in it.*

Definition: 1.6[11]

Let $(X, \|.\|_g)$ *be a* G*- normed space . for* $r > 0$ *we define*

$$B_g(x,r) = \{ y \in X: \|x-y\|_g < r \}$$

Let X *be a* G*- Normed space and* A *be a subset of* X*. Then for every* $x \in A$*, there exists* $r > 0$*, such that* $B_g(x,r) \subset A$*, then the subset* A *is called open subset of* X*. a subset* A *of* X *is said to be closed if the complement of* A *is open in* X*.*

Definition: 1.7[11]

A subset A *of* X *is said to be* G*- bounded if there exists* $r > 0$ *such that*

$$\|x-y\|_g < r \ for\ all\ x,y \in A.$$

Some example of $\|.\|_g$ *are as follows:*

a) let X *be a nonempty set then we define* $\|x-y\|_g = \|x-y\|$ *for every* $x,y \in X$*, Where* $a \,\nabla\, b = a+b$ *for* $a,b \in R^+$ *and* $\|.\|$ *is ordinary normed space on* X*.*

b) let X *be a non empty set. We define,*

$$\|x-y\|_g = \begin{cases} 0 \ , & x = y \\ 1 \ , otherwise \end{cases}$$

For each $x,y \in X$*, where* $a \,\nabla\, b = max\{a,b\}$ *for* $a,b \in R^+$*.*

Lemma: 1.9[11]

Let $(X, \|.\|_g)$ *be a* G*- Normed space such that,* ∇*, setisfy* $\alpha - property$ *with* $\alpha > 0$*.*
if sequence $\{x_n\}$ *in* X *is converges to* x*, then* x*, is unique.*

Lemma: 1.10[11]

Let $(X, \|.\|_g)$ *be a* G*- Normed space such that,* ∇*, setisfy* $\alpha - property$ *with* $\alpha > 0$*.*
if sequence $\{x_n\}$ *in* X *is converges to* x*, then* $\{x_n\}$ *is Cauchy sequence.*

Definition: 1.11[11]

Let A *and* S *be mappings from a* G*- Banach space* X *into itself. Then the mappings are said to be weakly compatible if they are commute at their coincidence point, that is* $Ax = Sx$ *implies that,* $ASx = SAx$*.*

II. Main Results

Theorem 2.1:- *Let* X *be a complete* G*- Banach space such that* ∇ *satisfy* $\alpha - property$ *with* $\alpha \le 1$*. If* T *be a mapping from* X *into it, satisfying the following condition;*

$$\|Tx - Ty\|_g \le k_1 \left(\frac{\|x-Tx\|_g \|x-Ty\|_g}{\|x-y\|_g} \nabla \frac{\|y-Tx\|_g \|y-Ty\|_g}{\|x-y\|_g} \right)$$

$$+ k_2 \left(\frac{\|x-Tx\|_g \|y-Ty\|_g}{\|x-y\|_g} \nabla \frac{\|x-Ty\|_g \|y-Tx\|_g}{\|x-y\|_g} \right)$$

$$+ k_3 \{ \|x-Tx\|_g \nabla \|y-Ty\|_g \nabla \|x-Ty\|_g \nabla \|y-Tx\|_g \nabla \|x-y\|_g \} \qquad 2.1.1$$

For non negative k_1, k_2, k_3 *such that* $0 < k_1 + k_2 + k_3 < 1$. *Then* T *has unique fixed point in* X.

Proof *:- let* x_0 *be arbitrary point in* X, *then we choose a point* x_1 *in* X *such that* $x_1 = Tx_0$. *In general we have a sequence* $\{x_n\}$ *in* X *such that,* $x_{n+1} = Tx_n$

Now $\|x_{n+1} - x_{n+2}\|_g = \|Tx_n - Tx_{n+1}\|_g$

From 2.1.1 we have,

$$\|Tx_n - Tx_{n+1}\|_g \leq k_1\left(\frac{\|x_n - Tx_n\|_g\|x_n - Tx_{n+1}\|_g}{\|x_n - x_{n+1}\|_g}\nabla\frac{\|x_{n+1} - Tx_n\|_g\|x_{n+1} - Tx_{n+1}\|_g}{\|x_n - x_{n+1}\|_g}\right)$$

$$+ k_2\left(\frac{\|x_n - Tx_n\|_g\|x_{n+1} - Tx_{n+1}\|_g}{\|x_n - x_{n+1}\|_g}\nabla\frac{\|x_n - Tx_{n+1}\|_g\|x_{n+1} - Tx_n\|_g}{\|x_n - x_{n+1}\|_g}\right)$$

$$+ k_3\begin{Bmatrix}\|x_n - Tx_n\|_g\nabla\|x_{n+1} - Tx_{n+1}\|_g\nabla\|x_n - Tx_{n+1}\|_g\\ \nabla\|x_{n+1} - Tx_n\|_g\nabla\|x_n - x_{n+1}\|_g\end{Bmatrix}$$

$$\|Tx_n - Tx_{n+1}\|_g \leq k_1\left(\frac{\|x_n - x_{n+1}\|_g\|x_n - x_{n+2}\|_g}{\|x_n - x_{n+1}\|_g}\nabla\frac{\|x_{n+1} - x_{n+1}\|_g\|x_{n+1} - x_{n+2}\|_g}{\|x_n - x_{n+1}\|_g}\right)$$

$$+ k_2\left(\frac{\|x_n - x_{n+1}\|_g\|x_{n+1} - x_{n+2}\|_g}{\|x_n - x_{n+1}\|_g}\nabla\frac{\|x_n - x_{n+2}\|_g\|x_{n+1} - x_n\|_g}{\|x_n - x_{n+1}\|_g}\right)$$

$$+ k_3\begin{Bmatrix}\|x_n - x_{n+1}\|_g\nabla\|x_{n+1} - x_{n+2}\|_g\nabla\|x_n - x_{n+2}\|_g\\ \nabla\|x_{n+1} - x_{n+1}\|_g\nabla\|x_n - x_{n+1}\|_g\end{Bmatrix}$$

$$\|x_{n+1} - x_{n+2}\|_g \leq k_1 max(\|x_n - x_{n+1}\|_g, \|x_{n+1} - x_{n+2}\|_g)$$
$$+ k_2 max(\|x_n - x_{n+1}\|_g, \|x_{n+1} - x_{n+2}\|_g)$$
$$+ k_3 max\{\|x_n - x_{n+1}\|_g, \|x_{n+1} - x_{n+2}\|_g\}$$

If we take max, $\|x_{n+1} - x_{n+2}\|_g$, *then we have,*

$$\|x_{n+1} - x_{n+2}\|_g \leq (k_1 + k_2 + k_3)\|x_{n+1} - x_{n+2}\|_g$$

Which contradiction the hypothesis, so we have

$$\|x_{n+1} - x_{n+2}\|_g \leq (k_1 + k_2 + k_3)\|x_n - x_{n+1}\|_g$$

Similarly we can find,

$$\|x_{n+1} - x_n\|_g \leq (k_1 + k_2 + k_3)\|x_{n-1} - x_n\|_g$$

In this way, we can write,

$$\|x_{n+1} - x_n\|_g \leq (k_1 + k_2 + k_3)^n\|x_{n-1} - x_n\|_g$$

$$\lim_{n\to\infty}\|x_{n+1} - x_n\|_g \to 0$$

which implies, $\{x_n\}$ *is a Cauchy sequence,. Which converges to* $'u'$ *in* X.

Now on taking, $x = x_n$ *and* $y = u$ *in* [2.1.1], *then we get*

$$\|Tx_n - Tu\|_g \leq k_1\left(\frac{\|x_n - Tx_n\|_g\|x_n - Tu\|_g}{\|x_n - u\|_g}\nabla\frac{\|u - Tx_n\|_g\|u - Tu\|_g}{\|x_n - u\|_g}\right)$$

$$+ k_2\left(\frac{\|x_n - Tx_n\|_g\|u - Tu\|_g}{\|x_n - u\|_g}\nabla\frac{\|x_n - Tu\|_g\|u - Tx_n\|_g}{\|x_n - u\|_g}\right)$$

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$$+ k_3 \begin{Bmatrix} \|x_n - Tx_n\|_g \nabla \|u - Tu\|_g \nabla \|x_n - Tu\|_g \\ \nabla \|u - Tx_n\|_g \nabla \|x_n - u\|_g \end{Bmatrix}$$

as $n \to \infty$, *we get,* $Tu = u$.

$i.e.\ u, is\ a\ fixed\ point\ of\ T\ in\ X.$

Uniqueness

Let us assume that, 'w' is another fixed point of T, different from 'u' in X. then, $u \neq w$

$$\|Tw - Tu\|_g \le k_1 \left(\frac{\|w - Tw\|_g \|w - Tu\|_g}{\|w - u\|_g} \nabla \frac{\|u - Tw\|_g \|u - Tu\|_g}{\|w - u\|_g} \right)$$

$$+ k_2 \left(\frac{\|w - Tw\|_g \|u - Tu\|_g}{\|w - u\|_g} \nabla \frac{\|w - Tu\|_g \|u - Tw\|_g}{\|w - u\|_g} \right)$$

$$+ k_3 \begin{Bmatrix} \|w - Tw\|_g \nabla \|u - Tu\|_g \nabla \|w - Tu\|_g \\ \nabla \|u - Tw\|_g \nabla \|w - u\|_g \end{Bmatrix}$$

$$\|u - w\| \le (k_1 + k_2 + k_3) \|u - w\|_g$$

This is a contradiction .so 'u' is unique fixed point of T, in X.

Theorem 2.2:-

Let X be a complete G- Banach space such that ∇ *satisfy* $\alpha - property$ *with* $\alpha \le 1$. *If S, T be compatible mapping from X into itself, satisfying the following condition;*

$$\|Sx - Ty\|_g \le k_1 \left(\frac{\|x - Sx\|_g \|x - Ty\|_g}{\|x - y\|_g} \nabla \frac{\|y - Sx\|_g \|y - Ty\|_g}{\|x - y\|_g} \right)$$

$$+ k_2 \left(\frac{\|x - Sx\|_g \|y - Ty\|_g}{\|x - y\|_g} \nabla \frac{\|x - Ty\|_g \|y - Sx\|_g}{\|x - y\|_g} \right)$$

$$+ k_3 \{ \|x - Sx\|_g \nabla \|y - Ty\|_g \nabla \|x - Ty\|_g \nabla \|y - Sx\|_g \nabla \|x - y\|_g \} \qquad 2.2.1$$

for non negative k_1, k_2, k_3 *such that* $0 < k_1 + k_2 + k_3 < 1$. *Then S , T have unique common fixed point in X.*

Proof :-

Let x_0 *be arbitrary point in X , then we choose a point* x_1 *in X such that* $x_1 = Tx_0$. *In general we have a sequence* $\{x_n\}$ *in X such that,* $x_{n+1} = Sx_n$, $x_{n+2} = Tx_{n+1}$

Now

$$\|x_{n+1} - x_{n+2}\|_g = \|Sx_n - Tx_{n+1}\|_g$$

From 2.2.1 *we have,*

$$\|Sx_n - Tx_{n+1}\|_g \le k_1 \left(\frac{\|x_n - Sx_n\|_g \|x_n - Tx_{n+1}\|_g}{\|x_n - x_{n+1}\|_g} \nabla \frac{\|x_{n+1} - Sx_n\|_g \|x_{n+1} - Tx_{n+1}\|_g}{\|x_n - x_{n+1}\|_g} \right)$$

$$+ k_2 \left(\frac{\|x_n - Sx_n\|_g \|x_{n+1} - Tx_{n+1}\|_g}{\|x_n - x_{n+1}\|_g} \nabla \frac{\|x_n - Tx_{n+1}\|_g \|x_{n+1} - Sx_n\|_g}{\|x_n - x_{n+1}\|_g} \right)$$

$$+ k_3 \begin{Bmatrix} \|x_n - Sx_n\|_g \nabla \|x_{n+1} - Tx_{n+1}\|_g \nabla \|x_n - Tx_{n+1}\|_g \\ \nabla \|x_{n+1} - Sx_n\|_g \nabla \|x_n - x_{n+1}\|_g \end{Bmatrix}$$

$$\|x_{n+1} - x_{n+2}\|_g \le k_1 \left(\frac{\|x_n - x_{n+1}\|_g \|x_n - x_{n+2}\|_g}{\|x_n - x_{n+1}\|_g} \nabla \frac{\|x_{n+1} - x_{n+1}\|_g \|x_{n+1} - x_{n+2}\|_g}{\|x_n - x_{n+1}\|_g} \right)$$

$$+ k_2 \left(\frac{\|x_n - x_{n+1}\|_g \|x_{n+1} - x_{n+2}\|_g}{\|x_n - x_{n+1}\|_g} \nabla \frac{\|x_n - x_{n+2}\|_g \|x_{n+1} - x_n\|_g}{\|x_n - x_{n+1}\|_g} \right)$$

$$+ k_3 \begin{Bmatrix} \|x_n - x_{n+1}\|_g \nabla \|x_{n+1} - x_{n+2}\|_g \nabla \|x_n - x_{n+2}\|_g \\ \nabla \|x_{n+1} - x_{n+1}\|_g \nabla \|x_n - x_{n+1}\|_g \end{Bmatrix}$$

$$\|x_{n+1} - x_{n+2}\|_g \le k_1 max(\|x_n - x_{n+1}\|_g, 0)$$
$$+ k_2 max(\|x_n - x_{n+1}\|_g, \|x_{n+1} - x_{n+2}\|_g)$$
$$+ k_3 max\{\|x_n - x_{n+1}\|_g, \|x_{n+1} - x_{n+2}\|_g\}$$

If we take max, $\|x_{n+1} - x_{n+2}\|_g$*, then we have,*

$$\|x_{n+1} - x_{n+2}\|_g \le (k_1 + k_2 + k_3)\|x_{n+1} - x_{n+2}\|_g$$

Which contradiction the hypothesis, so we have

$$\|x_{n+1} - x_{n+2}\|_g \le (k_1 + k_2 + k_3)\|x_n - x_{n+1}\|_g$$

Similarly we can find,

$$\|x_{n+1} - x_n\|_g \le (k_1 + k_2 + k_3)\|x_{n-1} - x_n\|_g$$

In this way, we can write,

$$\|x_{n+1} - x_n\|_g \le (k_1 + k_2 + k_3)^n \|x_{n-1} - x_n\|_g$$

$as \ n \to \infty$

$\lim_{n\to\infty}\|x_{n+1} - x_n\|_g \to 0$

Which implies, $\{x_n\}$ *is a Cauchy sequence,. Which converges to* $'u'$ *in X.*

Since S and T are compatible mapping, which implies,

$u = \lim_{n\to\infty} STx_n = \mathrm{S}\lim_{n\to\infty} Tx_n = Su$, *also*

$u = \lim_{n\to\infty} STx_n = \mathrm{T}\lim_{n\to\infty} Sx_n = Tu.$

i.e, 'u' is common fixed point of S and T, in X.

Uniqueness,

Let us assume that, 'w' is another fixed point of T, different from 'u' in X. then, $u \neq w$

$$\|Sw - Tu\|_g \le k_1\left(\frac{\|w - Sw\|_g\|w - Tu\|_g}{\|w - u\|_g}\nabla\frac{\|u - Sw\|_g\|u - Tu\|_g}{\|w - u\|_g}\right)$$
$$+ k_2\left(\frac{\|w - Sw\|_g\|u - Tu\|_g}{\|w - u\|_g}\nabla\frac{\|w - Tu\|_g\|u - Sw\|_g}{\|w - u\|_g}\right)$$
$$+ k_3\left\{\begin{matrix}\|w - Sw\|_g\nabla\|u - Tu\|_g\nabla\|w - Tu\|_g\\ \nabla\|u - Sw\|_g\nabla\|w - u\|_g\end{matrix}\right\}$$

$$\|u - w\| \le (k_1 + k_2 + k_3)\|u - w\|_g$$

This is a contradiction .So 'u' is unique common fixed point of S, and T, in X.

Theorem: 2.3 *Let X be a G- Banach space , such that* ∇ *satisfy property with* $\alpha - \alpha \le 1$*. If* A, B, S *and* T *be mapping from X into itself satisfying the following condition:*

i. $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, *and* $T(X)$ *or* $S(X)$ *is a closed subset of X.*
ii. *The pair* (A, S)*and* (B, T) *are weakly compatible,*
iii. *For all* $x, y \in X$,

$$\|Ax - By\|_g \le k_1\left(\frac{\|Sx-Ax\|_g\|Sx-By\|_g}{\|Sx-Ty\|_g}\nabla\frac{\|Ty-Ax\|_g\|Ty-By\|_g}{\|Sx-Ty\|_g}\right)$$
$$+ k_2\left(\frac{\|Sx - Ax\|_g\|Ty - By\|_g}{\|Sx - Ty\|_g}\nabla\frac{\|Sx - By\|_g\|Ty - Ax\|_g}{\|Sx - Ty\|_g}\right)$$
$$+ k_3\{\|Sx - Ax\|_g\nabla\|Ty - By\|_g\nabla\|Sx - By\|_g\nabla\|Ty - Ax\|_g\nabla\|Sx - Ty\|_g\}$$

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Where $k_1, k_2, k_3 > 0$ and $0 < k_1 + k_2 + k_3 < 1$. Then A, B, S, and T have a unique Common fixed point in X.

Proof :-

Let x_0 be an arbitrary point in X. then by (i), we choose a point x_1 in X such that, $y_0 = Ax_0 = Tx_1$ and $y_1 = Bx_1 = Sx_2$. In general, there exists a sequence $\{y_n\}$ such that, $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 1,2,3, \dots \dots$

We claim that the sequence $\{y_n\}$ is Cauchy sequence.

By (iii), we have,

$$\|y_{2n} - y_{2n+1}\|_g = \|Ax_{2n}, Bx_{2n+1}\|_g$$

$$\|Ax_{2n} - Bx_{2n+1}\|_g \le k_1 \left(\frac{\|Sx_{2n} - Ax_{2n}\|_g \|Sx_{2n} - Bx_{2n+1}\|_g}{\|Sx_{2n} - Tx_{2n+1}\|_g} \nabla \frac{\|Tx_{2n+1} - Ax_{2n}\|_g \|Tx_{2n+1} - Bx_{2n+1}\|_g}{\|Sx_{2n} - Tx_{2n+1}\|_g} \right)$$

$$+ k_2 \left(\frac{\|Sx_{2n} - Ax_{2n}\|_g \|Tx_{2n+1} - Bx_{2n+1}\|_g}{\|Sx_{2n} - Tx_{2n+1}\|_g} \nabla \frac{\|Sx_{2n} - Bx_{2n+1}\|_g \|Tx_{2n+1} - Ax_{2n}\|_g}{\|Sx_{2n} - Tx_{2n+1}\|_g} \right)$$

$$+ k_3 \left\{ \begin{matrix} \|Sx_{2n} - Ax_{2n}\|_g \nabla \|Tx_{2n+1} - Bx_{2n+1}\|_g \nabla \|Sx_{2n} - Bx_{2n+1}\|_g \\ \nabla \|Tx_{2n+1} - Ax_{2n}\|_g \nabla \|Sx_{2n} - Tx_{2n+1}\|_g \end{matrix} \right\}$$

$$\|y_{2n} - y_{2n+1}\|_g \le k_1 \left(\frac{\|y_{2n-1} - y_{2n}\|_g \|y_{2n} - y_{2n+1}\|_g}{\|y_{2n-1} - y_{2n}\|_g} \nabla \frac{\|y_{2n} - y_{2n}\|_g \|y_{2n} - y_{2n+1}\|_g}{\|y_{2n-1} - y_{2n}\|_g} \right)$$

$$+ k_2 \left(\frac{\|y_{2n-1} - y_{2n}\|_g \|y_{2n} - y_{2n+1}\|_g}{\|y_{2n-1} - y_{2n}\|_g} \nabla \frac{\|y_{2n-1} - y_{2n+1}\|_g \|y_{2n} - y_{2n}\|_g}{\|y_{2n-1} - y_{2n}\|_g} \right)$$

$$+ k_3 \left\{ \begin{matrix} \|y_{2n-1} - y_{2n}\|_g \nabla \|y_{2n} - y_{2n+1}\|_g \nabla \|y_{2n-1} - y_{2n+1}\|_g \\ \nabla \|y_{2n} - y_{2n}\|_g \nabla \|y_{2n-1} - y_{2n}\|_g \end{matrix} \right\}$$

$$\|y_{2n} - y_{2n+1}\|_g \le k_1 max \left(\frac{\|y_{2n-1} - y_{2n}\|_g \|y_{2n} - y_{2n+1}\|_g}{\|y_{2n-1} - y_{2n}\|_g} \nabla \frac{\|y_{2n} - y_{2n}\|_g \|y_{2n} - y_{2n+1}\|_g}{\|y_{2n-1} - y_{2n}\|_g} \right)$$

$$+ k_2 \left(\frac{\|y_{2n-1} - y_{2n}\|_g \|y_{2n} - y_{2n+1}\|_g}{\|y_{2n-1} - y_{2n}\|_g} \nabla \frac{\|y_{2n-1} - y_{2n+1}\|_g \|y_{2n} - y_{2n}\|_g}{\|y_{2n-1} - y_{2n}\|_g} \right)$$

$$+ k_3 \left\{ \begin{matrix} \|y_{2n-1} - y_{2n}\|_g \nabla \|y_{2n} - y_{2n+1}\|_g \nabla \|y_{2n-1} - y_{2n+1}\|_g \\ \nabla \|y_{2n} - y_{2n}\|_g \nabla \|y_{2n-1} - y_{2n}\|_g \end{matrix} \right\}$$

$$\|y_{2n} - y_{2n+1}\|_g \le (k_1 + k_2 + k_3) \|y_{2n-1} - y_{2n}\|_g$$

That is by induction we can show that

$$\|y_{2n} - y_{2n+1}\|_g \le (k_1 + k_2 + k_3)^n \|y_0 - y_1\|_g$$

As $n \to \infty$, $\|y_{2n} - y_{2n+1}\|_g \to 0$

For any integer $m \ge n$

It follows that, the sequence $\{y_n\}$ is a Cauchy sequence which converges to $y \in X$.

This implies that

$$lim_{n\to\infty} y_n = lim_{n\to\infty} Ax_{2n} = lim_{n\to\infty} Bx_{2n+1} = lim_{n\to\infty} Sx_{2n+2} = lim_{n\to\infty} Tx_{2n+1} = y$$

Now let us assume that, $T(X)$ is closed subset of X, then there exists $v \in X$ such that

$Tv = y$. We now prove that $Bv = y$. by (iii), we get

$$\|Ax_{2n}-Bv\|_g \leq k_1\left(\frac{\|Sx_{2n}-Ax_{2n}\|_g\|Sx_{2n}-Bv\|_g}{\|Sx_{2n}-Tv\|_g}\nabla\frac{\|Tv-Ax_{2n}\|_g\|Tv-Bv\|_g}{\|Sx_{2n}-Tv\|_g}\right)$$

$$+k_2\left(\frac{\|Sx_{2n}-Ax_{2n}\|_g\|Tv-Bv\|_g}{\|Sx_{2n}-Tv\|_g}\nabla\frac{\|Sx_{2n}-Bv\|_g\|Tv-Ax_{2n}\|_g}{\|Sx_{2n}-Tv\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|Sx_{2n}-Ax_{2n}\|_g\nabla\|Tv-Bv\|_g\nabla\|Sx_{2n}-Bv\|_g\\ \nabla\|Tv-Ax_{2n}\|_g\nabla\|Sx_{2n}-Tv\|_g\end{Bmatrix}$$

$$as\ n\to\infty,\quad \|y-Bv\|_g \leq (k_1+k_2+k_3)\ \|y-Bv\|_g$$

Which contradiction, it follows that $Bv = y = Tv$. *Since* B *and* T *are weakly compatible mappings, then we have* $BTv = TBv$ *which implies* $By = Ty$.

Now we prove that, $By = y$, *for this by using (iii), we get*

$$\|Ax_{2n}-By\|_g \leq k_1\left(\frac{\|Sx_{2n}-Ax_{2n}\|_g\|Sx_{2n}-By\|_g}{\|Sx_{2n}-Ty\|_g}\nabla\frac{\|Ty-Ax_{2n}\|_g\|Ty-By\|_g}{\|Sx_{2n}-Ty\|_g}\right)$$

$$+k_2\left(\frac{\|Sx_{2n}-Ax_{2n}\|_g\|Ty-By\|_g}{\|Sx_{2n}-Ty\|_g}\nabla\frac{\|Sx_{2n}-By\|_g\|Ty-Ax_{2n}\|_g}{\|Sx_{2n}-Ty\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|Sx_{2n}-Ax_{2n}\|_g\nabla\|Ty-By\|_g\nabla\|Sx_{2n}-By\|_g\\ \nabla\|Ty-Ax_{2n}\|_g\nabla\|Sx_{2n}-Ty\|_g\end{Bmatrix}$$

$$as\ n\to\infty,\quad \|y-By\|_g \leq (k_1+k_2+k_3)\ \|y-By\|_g$$

Which contradiction. Thus $By = y = Ty$

Since $B(X) \subseteq S(X)$, *there exists* $w \in X$. *such that* $Sw = y$. *we show that,* $Aw = y$. *from (iii)*

$$\|Aw-By\|_g \leq k_1\left(\frac{\|Sw-Aw\|_g\|Sw-By\|_g}{\|Sw-Ty\|_g}\nabla\frac{\|Ty-Aw\|_g\|Ty-By\|_g}{\|Sw-Ty\|_g}\right)$$

$$+k_2\left(\frac{\|Sw-Aw\|_g\|Ty-By\|_g}{\|Sw-Ty\|_g}\nabla\frac{\|Sw-By\|_g\|Ty-Aw\|_g}{\|Sw-Ty\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|Sw-Aw\|_g\nabla\|Ty-By\|_g\nabla\|Sw-By\|_g\\ \nabla\|Ty-Aw\|_g\nabla\|Sw-Ty\|_g\end{Bmatrix}$$

$$\|Aw-By\|_g \leq (k_1+k_2+k_3)\ \|Aw-By\|_g$$

$$\|Aw-y\|_g \leq (k_1+k_2+k_3)\ \|Aw-y\|_g$$

Which contradiction, so that $Aw = y = Sw$ *Since* A *and* S *are weakly compatible, then* $ASw = SAw$ *and so* $Ay = Sy$.

Now we show that, $Ay = y$, *from (iii),*

$$\|Ay-By\|_g \leq k_1\left(\frac{\|Sy-Ay\|_g\|Sy-By\|_g}{\|Sy-Ty\|_g}\nabla\frac{\|Ty-Ay\|_g\|Ty-By\|_g}{\|Sy-Ty\|_g}\right)$$

$$+k_2\left(\frac{\|Sy-Ay\|_g\|Ty-By\|_g}{\|Sy-Ty\|_g}\nabla\frac{\|Sy-By\|_g\|Ty-Ay\|_g}{\|Sy-Ty\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|Sy-Ay\|_g\nabla\|Ty-By\|_g\nabla\|Sy-By\|_g\\ \nabla\|Ty-Ay\|_g\nabla\|Sy-Ty\|_g\end{Bmatrix}$$

$$\|Ay-y\|_g \leq (k_1+k_2+k_3)\ \|Ay-y\|_g$$

Which contradiction,

Thus $Ay = y$ *and therefore* $Ay = Sy = By = Ty = y$.

y is a common fixed point of A, B, S, T, in X.

The proof is similar when we assume that, $S(X)$ is a closed subset of X

Uniqueness:-

Let us assume that x is another fixed point of A, B, S, T different from y in X.

Then from (iii), we have

$$\|Ax - By\|_g \leq k_1 \left(\frac{\|Sx-Ax\|_g \|Sx-By\|_g}{\|Sx-Ty\|_g} \nabla \frac{\|Ty-Ax\|_g \|Ty-By\|_g}{\|Sx-Ty\|_g} \right)$$

$$+ k_2 \left(\frac{\|Sx - Ax\|_g \|Ty - By\|_g}{\|Sx - Ty\|_g} \nabla \frac{\|Sx - By\|_g \|Ty - Ax\|_g}{\|Sx - Ty\|_g} \right)$$

$$+ k_3 \left\{ \begin{matrix} \|Sx - Ax\|_g \nabla \|Ty - By\|_g \nabla \|Sx - By\|_g \\ \nabla \|Ty - Ax\|_g \nabla \|Sx - Ty\|_g \end{matrix} \right\}$$

$$\|x - y\|_g \leq (k_1 + k_2 + k_3)\ \|x - y\|_g$$

This is contradiction. Thus $x = y$. This completes the proof of the theorem.

Remark:-

1. *If we take $S = T$ in theorem- 2.2 then we get theorem 2.1*
2. *If we take $S = T = I$ in theorem- 2.3 then we get theorem 2.2*
3. *If we take $A = B$ and $S = T = I$ in theorem - 2.3 then we get theorem 2.1*

Corollary 2.4 *: Let X be a complete G- Banach space such that ∇ satisfy $\alpha - property$ with $\alpha \leq 1$. If T be a mapping from X into it, satisfying the following condition;*

$$\|T^r x - T^s y\|_g \leq k_1 \left(\frac{\|x - T^r x\|_g \|x - T^s y\|_g}{\|x - y\|_g} \nabla \frac{\|y - T^r x\|_g \|y - T^s y\|_g}{\|x - y\|_g} \right)$$

$$+ k_2 \left(\frac{\|x - T^r x\|_g \|y - T^s y\|_g}{\|x - y\|_g} \nabla \frac{\|x - T^s y\|_g \|y - T^r x\|_g}{\|x - y\|_g} \right)$$

$$+ k_3 \{ \|x - T^r x\|_g \nabla \|y - T^s y\|_g \nabla \|x - T^s y\|_g \nabla \|y - T^r x\|_g \nabla \|x - y\|_g \} \qquad 2.4.1$$

For non negative k_1, k_2, k_3 such that $0 < k_1 + k_2 + k_3 < 1$, and $r, s \in N(set\ of\ natural\ number)$. Then T has unique fixed point in X.

Proof : This can be proved easily by theorem – 2.1, on taking $r = s = 1$.

Corollary 2.5*: Let X be a complete G- Banach space such that ∇ satisfy $\alpha - property$ with $\alpha \leq 1$. If S, T be compatible mapping from X into itself, satisfying the following condition;*

$$\|S^r x - T^u y\|_g \leq k_1 \left(\frac{\|x - S^r x\|_g \|x - T^u y\|_g}{\|x - y\|_g} \nabla \frac{\|y - S^r x\|_g \|y - T^u y\|_g}{\|x - y\|_g} \right)$$

$$+ k_2 \left(\frac{\|x - S^r x\|_g \|y - T^u y\|_g}{\|x - y\|_g} \nabla \frac{\|x - T^u y\|_g \|y - S^r x\|_g}{\|x - y\|_g} \right)$$

$$+ k_3 \{ \|x - S^r x\|_g \nabla \|y - T^u y\|_g \nabla \|x - T^u y\|_g \nabla \|y - S^r x\|_g \nabla \|x - y\|_g \} \qquad 2.2.1$$

For non negative k_1, k_2, k_3 such that $0 < k_1 + k_2 + k_3 < 1$. and $r, u \in N$ (set of natural number) Then S, T have unique common fixed point in X.

Proof : *This can be proved easily by theorem – 2.2, on taking $r = u = 1$.*

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